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PERIODIC ORBITS ABOUT THE LAGRANGIAN STABLE POINTS OF THE EARTH - MOON SYSTEM

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Abstract. *Periodic solutions in the neighborhood of the Lagrangian Stable Equilibrium Points for the Elliptic Planar Restricted Three-Body problem are developed by a convergent method of successive approximations.*

Keywords: *Three-Body, Restricted, Elliptic, Periodic, Orbits*

1. INTRODUCTION

Motion in the vicinity of the Earth-Moon system is particularly important for two reasons: space exploration missions to the Moon and parking orbits in the vicinity of this system, for observation purposes and space surveillance. The present problem corresponds to the actual determination of initial conditions leading to periodic solutions in the vicinity of the equilateral equilibrium points of the above system. The elliptic restricted problem of three bodies, and the stability properties of the Lagrangian solutions, was studied in details by Giacaglia and Szebehely (1969). A convergent method of obtaining the characteristic exponents at L_4 and L_5, in the elliptic case, was presented by Giacaglia (1971), who also dedicated several papers to the resonace cases (Giacaglia 1968; Giacaglia, 1969; Giacaglia and França, 1970; Giacaglia and Nacozy, 1970) The elliptic case has received little attention, although for large eccentricity of the primaries, should give important informations about the evolution of bodies gravitating in the field of disrupting binary systems. Recent important work on the restricted problem, have shown this lack of interest. In a recent paper, Winter and Murray (1997) have integrated periodic symmetric librations associated with the 1:n resonances for n = 2,3,4,5. They also discuss the existence and properties of asymmetric librations for large values of n. Chaotic behavior of resonant orbits about the triangular equilibrium points was observed by numerical integration. It may be interesting to see the influence of the eccentricity of the primaries in such behavior. Laskar and Robutel (1995) presented the notion that has lately been introduced about the chaotic behavior of the solar system, established by numerical integration. Their work gives a new way of expanding the Hamiltonian of the planetary system, putting in evidence the ratios of all the semi-major axis. The applicability of Arnold and Kolmogorov Theorems is also given attention. In the context of their work, the important point is the chaotic behavior, although extensive numerical integrations of the restricted problem by Broucke(1971), Giacaglia and Nacozy (1970), Giacaglia (1968) and more recently by Ragos et al. (1997), have not demonstrated such behavior. Robutel (1995) uses a computer algebraic manipulator to expand the Hamiltonian of the three-body problem and uses the results to demonstrate the existence of quasi-periodic orbits, as predicted by Arnold Theorem. Boundedness of the librations about the triangular points is obtained by an analytical procedure which eliminates fast variables by Hagel (1996), concluding with the comparison and agreement with numerical integration. Chaotic behavior is not put in evidence, for orbits in the vicinity of the 1:2 resonance. An interesting analysis is performed by Celletti and Ferrara (1996) showing the stability of the restricted problem configuration in the special case Sun-Ceres-Jupiter for a very long time, comparable with the age of the solar system. Except for a few mass ratios, Ishwar (1997) demonstrates the stability of L_4 even in the case of a small oblateness of one of the primaries and of the infinitesimal body. Again, it might be of interest to examine the influence of the eccentricity of the primaries in the above results. Similar studies have been carried out by Subba Rao and Sharma (1997). Thakur ans Singh (1997) analyze a different generalization of the restricted problem, introducing radiation of one of the primaries and oblateness of the other. They analyze the stability of the triangular points in the presence of resonances 1:2 and 1:3. Instability is shown in the 1:2 case, while in the 1:3 case it depends on the radiation and oblateness parameters. Nothing is mentioned about the influence of eccentricity. If the eccentricity should change due to presence of radiation, oblateness, tides and other forces, over a long period the influence on stability may prove to be quite important. An extreme case of eccentricity is considered by Broucke (1971) and Martinez Alfaro and Orellana (1997).

2. EQUATIONS OF THE PROBLEM

For the problem of the motion of an infinitesimal body in the gravitational field of two large masses in relative keplerian elliptic motion, a method for the determination of characteristic exponents has been proposed by this author (Giacaglia, 1971). Periodic oscillations about the triangular equilibrium points may also be developed by this method. In the paper referred above, it has been shown that the equations of motion in the vicinity of the Lagrangean points L_4 and L_5, in the elliptic restricted problem of three bodies, may be written as

$$\dot{z} = A\,z + \varepsilon\,\Phi\,z \tag{2.1}$$

where

$$z = (z_1, z_2, z_3, z_4),\ \ A = diag(\rho_1, \rho_2, \rho_3, \rho_4),\ \ \rho_j = \lambda_j - \varepsilon\sigma K_{jj}$$
$$\lambda_1 = -\lambda_3 = \sqrt{-(1+s)/2},\quad \lambda_2 = -\lambda_4 = \sqrt{-(1-s)/2}\ \text{(imaginary)}$$
$$s = \sqrt{1-27\mu(1-\mu)}, \varepsilon\sigma = 1-(1-\varepsilon^2)^{-1/2}, K_{jk}\ \text{are given by Eqs.40 in Giacaglia (1971)}$$
$$\mathrm{K}_{11} = -K_{33}, K_{22} = -K_{44}, \Phi = \{\Phi_{ij}\}\ \Phi_{ij} = (\sigma\delta_{ij} - \cos t(1+\varepsilon\cos t)^{-1})K_{ij}$$
$$\langle\Phi_{jj}\rangle = 0,\ \dot{z} = dz/dt,\ t = \text{true anomaly of the primaries}$$

and where

$$\langle F(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T F(t)\,dt,\quad \Phi_{ij}(t+2\pi) = \Phi_{ij}(t),\qquad P = 2\pi\ \text{ is the period}$$

Defining $\alpha = 3/4$, $\beta = 3\sqrt{3}(2\mu-1)/4$ and $\gamma = 9/4$, the original equations for the linearized problem are

$$\begin{aligned} \xi'' - 2\eta' &= \mathrm{F}(\alpha\xi + \beta\eta) \\ \eta'' - 2\xi' &= \mathrm{F}(\beta\xi + \gamma\eta) \end{aligned} \tag{2.2}$$

With the new variables $\zeta_1 = \xi$, $\zeta_2 = \eta, \xi' = \zeta_3, \xi' = \zeta_3, \eta' = \zeta_3$, system (2.2) takes the form

$$\zeta' = \mathrm{A}^*\zeta + \varepsilon\ \Phi^*\zeta \tag{2.3}$$

where

$$A^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & \beta & 0 & 2 \\ \beta & \gamma & -2 & 0 \end{pmatrix}, \ \Phi^* = -\frac{\cos f}{1+\varepsilon \ \cos f}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \end{pmatrix}, \text{ or, } \Phi^* = -\frac{\cos f}{1+\varepsilon \ \cos f}K^*$$

The characteristic roots of A^* are $\lambda_1 = -\lambda_3 = \sqrt{-(1+S)/2}$, $\lambda_2 = -\lambda_4 = \sqrt{-(1-S)/2}$, with $S = \sqrt{1-27\mu\ (1-\mu)}$. They are pure imaginary if $27\mu(\mu-1) \le 1$ i.e. $\mu \le \mu_0 = 0{,}03852\cdots$. They are different unless $S = 0$, i. e. , $\mu = \mu_0$. They are commensurable whenever $\mu(\mu-1) = 4n^2m^2(n^2+m^2)^{-2}/27$ with n, m natural numbers. We exclude the above singular cases. There exists C, $|C| \ne 0$, such that $C^{-1}A^*C = \tilde{A} = \mathrm{diag}(\lambda_i)$. For instance, $C_{1k} = 1$, $C_{2k} = \dfrac{\lambda_k^2 - \alpha}{\beta + 2\lambda_k} = \dfrac{\beta - 2\lambda_k}{\lambda_k^2 - \gamma}$, $C_{3k} = \lambda_k$, $C_{4k} = C_{2k}\lambda_k$, $k = (1,2,3,4)$. In this case, $|C| = N/D$ where $N = 4\beta^4 - (55/4 - 8\lambda_1\lambda_2)\beta^2 + (45/4 - 12\lambda_1\lambda_2)$ and $D = (\beta^2-3)(\beta^2-9)$. Now, $D = 0$ iff $\beta^2 = 3$ or $9 \Rightarrow \mu > 1$ and $N = 0$ iff $\beta^2 = 3 \Rightarrow \mu > 1$ or $\beta^2 = 23/16 \Rightarrow \mu > \mu_0$.
Let $\zeta = Cz$, $z' = C^{-1}A^*C\,z - \dfrac{\varepsilon \cos f}{1+\varepsilon\cos f}C^{-1}K^*C\,z$ or, defining $\tilde{A} = C^{-1}A^*C$, $K = C^{-1}K^*C$, we obtain the equation $z' = \tilde{A}\,z - \dfrac{\varepsilon \cos f}{1+\varepsilon\cos f}K\,z = \tilde{A}\,z + \varepsilon\,\tilde{\Phi}\,z$.

Let
$$\sigma = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos f}{1+\varepsilon\cos f}\,d f = \frac{1}{\varepsilon}\left(1 - \frac{1}{\sqrt{1-\varepsilon^2}}\right) = -\frac{1}{2}\varepsilon - \frac{3}{8}\varepsilon^3 - \cdots$$

Define $\Phi = \{\Phi_{ij}\}$, $\Phi_{ij} = \tilde{\Phi}_{ij} + \sigma\,K_{ij}\delta_{ij}$, $A = \{A_{ij}\}$, $A_{ij} = \tilde{A}_{ij} - \varepsilon\sigma\ K_{ij}\delta_{ij}$, $\rho_j = \lambda_j - \varepsilon\,\sigma\,K_{ij}$. The final form of the equations is given by Eq (2.1) above, with associated definitions. With the proposed method, to any order of approximation, one gets divisors $(\lambda_j - \lambda_k)^2 + \ell^2$, $\ell = 1,2,\cdots$, and parametric instability with respect to μ occurs when $\lambda_j - \lambda_k = \pm i\,\ell$. The following cases are possible

1) $\begin{matrix} j=1,\ k=3 \\ j=2,\ k=4 \end{matrix} \quad \to \lambda_1 = \lambda_2 = \pm i\,\dfrac{\ell}{2}$, which has been excluded

2) I) $27\mu(1-\mu) = 1 - \dfrac{1}{4}(\ell^2-2)^2$, $\ell \le 2$ if $0 \le \mu \le 1$

II) $27\mu(1-\mu) = 1 - \ell^2(2-\ell^2)$, $\ell = 1 \to \mu = 0{,}1 \quad \forall\ell \to |\mu| < 1$

I) $\ell = 1 \to 27\mu(1-\mu) = \dfrac{3}{4}$, $\mu = \dfrac{1}{2} \pm \dfrac{\sqrt{2}}{3} = \begin{cases} 0{,}9714 \\ 0{,}0286 < \mu_0 \end{cases}$, $\ell = 2 \to \mu = 0{,}1$

II) $\ell = 2 \to \mu = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{6} = \begin{cases} 0{,}7887 \\ 0{,}2113 \end{cases}$, none $< \mu_0$, $\ell > 2 \to \mu$ complex

In the interval $[0, \mu_0]$, parametric instability occurs only at $\mu^* = 0{,}0286$, a value not corresponding to the earth-moon system, where μ = 0.0123.

3. CONSTRUCTION OF PERIODIC SOLUTIONS

It is proposed here to construct periodic solutions of Eq. (2.1) by a convergent method of approximation. This method produces a contraction mapping in the Banach space of all periodic functions with minimum period *P'*. If such periodic solution exists, one may expect its period to be $P' = mP$, $m \neq 0$, integer, positive. Consider Eq.(2.1) written in the form

$$\dot{z} = B\,z + \varepsilon\,U\,z \tag{3.1}$$

where $\varepsilon U = \varepsilon\Phi + A - B$ and $B = diag(i\tau_1, i\tau_2, \ldots, i\tau_n)$. One has to assume that

$$\rho_j = i\tau_j + O(\varepsilon) = \rho_j(\varepsilon), \quad \tau_j = k_j / m_j \tag{3.2}$$

where k_j, m_j are integers, relative primes, $m_j > 0$ or $m_j = 1$ and $k_j = 0$, with $P' = mP$, $m = m_1 m_2 m_3 m_4$.

Successive approximations will produce a periodic solution of a system

$$\dot{y} = By + \varepsilon Uy - \varepsilon\, e^{Bt} f(\alpha;\tau;\rho;\varepsilon) \tag{3.3}$$

where $f = (f_1, f_2, f_3, f_4)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a constant vector, $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$ and $\rho = (\rho_1, \rho_2, \rho_3, \rho_4)$. Such solution is a periodic solution of Eq.(2.1) if it is possible to choose α and/or ρ such that

$$f_j(\alpha;\tau;\rho;\varepsilon) = 0 \tag{3.4}$$

can be satisfied for *j*=1, 2, 3, 4. Vector $f(\alpha;\tau;\rho;\varepsilon)$ is obtained by an averaging process necessary to avoid secular terms in the method of successive approximations. The linear non singular transformation

$$z = e^{Bt} u \tag{3.5}$$

reduces system (3.1) to

$$\dot{u} = \varepsilon V u \tag{3.6}$$

where

$$V = e^{-Bt} U\, e^{Bt} = V(t, \varepsilon) \tag{3.7}$$

Obviously, since τ_j are rational numbers, matrix *V* is periodic with period $P' = 2\pi\, m_1 m_2 m_3 m_4$. In this case, or for any function $F(t) = F(t + P')$, integrable in (0,*P'*), it follows that

$$M_t\, F(t) \equiv \langle F(t) \rangle = \frac{1}{P'} \int_0^{P'} F(t)\, dt \tag{3.8}$$

which is the constant term of Fourier series for *F*(*t*). For the integral of a function *G*(*t*) such that $\langle G(t) \rangle = 0$, we shall indicate the unique primitive function

$$H(t) = \int G(t)\, dt \tag{3.9}$$

such that $M_t H(t)=0$, which is periodic of period P' and absolutely continuous, provided $G(t)$ is at least L-integrable in $(0,P')$. It can be shown (Cesari, 1963), under these circumstances, that

$$|H(t)| \leq \frac{K}{P'}\int_0^{P'} |G(t)|\, dt \tag{3.10}$$

for some constant K independent of $G(t)$. It is also verified that $MF(t) \leq \max|(F(t)|$ and $|(I-M)F(t)| \equiv |F(t) - MF(t)| \leq 2\max|F(t)|$. These properties are necessary to show the convergence of the method to be used (Cesari, 1963, p. 125). We shall be using M in place of M_t with no danger of misunderstanding. The method of successive approximations is as follows. Consider Eq.(3.6) and define the following approximations

$$u^{(0)} = \alpha,\, u^{(m)} = u^{(0)} + \varepsilon\int (I-M)V(t,\varepsilon)u^{(m-1)}(t)\, dt \tag{3.11}$$

for m=1,2,3,.... It can be shown (Hale, 1963) that if $\|\alpha\| < r,\ 0 < r < R$, there exists an $\varepsilon_1,\ 0 < \varepsilon_1 < \varepsilon_0$, such that for all $|\varepsilon| < \varepsilon_1$, functions $u^{(m)}(t)$ are periodic with period P', mean value $Mu^{(m)} = \alpha$ and $\|u^{(m)}\| \leq R$. Also, $u^{(m)}(t)$ converges uniformly in $[0, P']$ as $m \to \infty$ to an absolutely continuous function $u(t)$ which is periodic with period P', $Mu(t) = \alpha, \|u(t)\| \leq R$ and $u(t)$ satisfies the integral equation

$$u = \alpha + \varepsilon\int (I-M)\, V(t,\varepsilon)\, u(t)\, dt \tag{3.12}$$

and, therefore, the differential equation

$$\dot{u} = \varepsilon V(t,\varepsilon)\, u - \varepsilon\, M[V(t,\varepsilon)\, u(t)] = \varepsilon V(t,\varepsilon)\, u - \varepsilon\, f(\alpha;\tau;\rho;\varepsilon) \tag{3.13}$$

Thus, Eq. (3.12) is a solution (periodic, with period P') of Eq. (3.6) if and only if

$$f(\alpha;\tau;\rho;\varepsilon) = 0 \tag{3.14}$$

for some vectors α and/or τ. The convergence of the process is established as follows. Consider Eq.(3.6) and suppose that the conditions $\|V(t)u(t)\| \leq L$, $\|V(t)u(t) - V(t)v(t)\| \leq L\|u(t) - v(t)\|$ are satisfied for all $t,\ |\varepsilon| \leq \varepsilon_0,\ \|u(t)\|\ and\ \|v(t)\| \leq R$. Consider the sequence defined by Eq.(12), given R, L, ε_0 and if $0 < r < R,\ \|\alpha\| \leq r$. If ***S*** is the space of all vector periodic functions $g(t)$ with period $T=2\pi / \omega$, continuous in $(-\infty,+\infty)$ with norm given by $Ng = \max\|g(t)\|$ for all t, consider the operator $Og = \alpha + \varepsilon\int (I-M)\, V(\tau)\, g(\tau)\, d\tau$ and let ***S**** be the set of all $g(t) \in$ ***S*** defined by ***S**** = $\{g \in$ ***S***, $Mg = \alpha,\ N g \leq R\}$. Then one has the mapping *O:**S**** $\Rightarrow$ ***S***. The following conditions are also satisfied

$$N(Og) \leq \|\alpha\| + |\varepsilon| KT^{-1}\int_0^T \|(I-M)V(\tau)g(\tau)\|d\tau \ \leq \|\alpha\| + 2|\varepsilon| KT^{-1}\int_0^T \|V(\tau)g(\tau)\|d\tau \leq \|\alpha\| + 2|\varepsilon| KL$$

so that one has $N(Og) \leq r + (R-r) = R$ for $|\varepsilon| \leq \varepsilon_1 = \min\{\varepsilon_0, (R-r)/2KL\}$. Also, $M(Og) = \alpha$, so that $O{:}S^* \Rightarrow S^*$ for $|\varepsilon| \leq \varepsilon_1$. Given $g, h \in S^*$, it follows that $N(Og - Oh) \leq 2|\varepsilon| KLN(g-h) \leq N(g-h)/2$, for $|\varepsilon| \leq \varepsilon_2 = min\{\varepsilon_1, 1/4KL\}$. Therefore ***O:S**** $\Rightarrow$ ***S**** is a contraction mapping. Space ***S**** is obviously complete, so that by Banach's fixed point theorem (Hale, 1963, p. 108), there is a unique element of ***S****, say *y(t)*$\in$ ***S****, such that $Oy=y$ and $N(y^{(m)}-y)\to 0$ as $m\to\infty$. Also $Oy=y$ implies that $y(t)=\alpha+\varepsilon\int(I-M)V(\tau)\,y(\tau)\,d\tau$ so that *y(t)* is absolutely continuous and $\dot{y}=\varepsilon Vy+\varepsilon D,\ D=M(Vy)$. The solution of the original system is established by using the inverse transformation

$$u=e^{-Bt}y \tag{3.15}$$

The successive approximations for system (3.3) are given by $y^{(0)}=e^{Bt}\alpha$ and , in general,

$$\begin{aligned} y^{(m)} &= e^{Bt}y^{(0)}+\varepsilon\, e^{Bt}\int(I-M)\,V(\tau,\varepsilon)\,e^{-B\tau}y^{(m-1)}(\tau)\,d\tau \\ &= e^{Bt}y^{(0)}+\varepsilon\int(I-M)\,e^{B(t-\tau)}\,U\,y^{(m-1)}(\tau)\,d\tau \\ &= e^{Bt}y^{(0)}+\int(I-M)\,e^{B(t-\tau)}\big[\varepsilon\,\Phi(\tau;\varepsilon)+A-B\big]\,y^{(m-1)}(\tau)\,d\tau \end{aligned} \tag{3.16}$$

The method converges to a function *y(t)* satisfying the integral equation

$$y(t)=e^{Bt}\alpha+\int(I-M)\,e^{B(t-\tau)}\big[\varepsilon\,\Phi(\tau,\varepsilon)+A-B\big]\,y(\tau)\,d\tau \tag{3.17}$$

or the differential equation

$$\begin{aligned} \dot{y} &= B\,y+e^{Bt}(I-M)\big[e^{-Bt}(\varepsilon\,\Phi+A-B)\,y\big]= \\ &= B\,y+(\varepsilon\,\Phi+A-B)\,y-e^{Bt}M\big[e^{-Bt}(\varepsilon\,\Phi+A-B)\,y\big] \\ &= B\,y+\varepsilon\,U\,y-\varepsilon\,e^{Bt}f(\alpha;\tau;\rho;\varepsilon) \end{aligned} \tag{3.18}$$

which is Eq.(3.3), with the definition

$$\varepsilon\,f(\alpha;\tau;\rho;\varepsilon)=M\big[e^{-Bt}(\varepsilon\Phi+A-B)y(t)\big]=\frac{1}{P'}\int_0^{P'}e^{-Bt}\big[\ \varepsilon\Phi(t)+A-B)\ \big]y(t)dt \tag{3.19}$$

For the solution to be possible , one should be able to solve for α the equation $f(\alpha;\tau;\rho;\varepsilon)=0$, so that it is necessary that the Jacobian $\left.\dfrac{\partial\, f(\alpha;\tau;\rho;0)}{\partial\,\alpha}\right|_{\alpha=\alpha_0}\neq 0$ so as to warrant the solution $\alpha=\alpha_0+\varepsilon\,g(\tau;\rho;\varepsilon)$ for some value α_0 of α. Conditionally periodic solutions may also be established, depending on special conditions. It has been assumed (Eq. 3.2) that all eigenvalues $\rho_j\ (j=1,2,3,4)$ are rational or close to rational numbers, in the sense of ε small. This is not the case for general values of the mass ratio μ. In fact one can choose μ so that $\lambda_1,\lambda_3(=-\lambda_1)$ are rational but not $\lambda_2,\lambda_4\ (=-\lambda_2)$, or vice-versa. This choice defines linear approximations (for ε = 0) which are short or long periodic, if one excludes the other pair of roots by proper choice of the initial conditions. Under these circumstances, it is possible to produce periodic solutions emanating from the short or long periodic approximations, for $\varepsilon\neq 0$. The above method of successive approximations applies equally well with a proper change in the choice of the initial vector α and matrix *B*. Let us consider the case when $\lambda_1,\lambda_3(=-\lambda_1)$ are rational but not $\lambda_2,\lambda_4\ (=-\lambda_2)$. That is, more precisely,

$$\rho_j(0)=\lambda_j=ik_j/m_j \qquad (j=1,3) \tag{3.20}$$

and

$$\rho_j(0) = \lambda_j \neq ip/m_1 m_2 \qquad (j = 2,4;\ p = 0,\pm 1,\pm 2,...) \tag{3.21}$$

It follows that

$$\rho_j = ik_j/m_j + O(\varepsilon) \quad (j = 1,3) \tag{3.22}$$

and the inequalities (3.21) hold for $\rho_j(\varepsilon)$, $j = 2, 4$, for $|\varepsilon|$ sufficiently small. Matrix B is now chosen as $B = diag(i\tau_1, \rho_2, i\tau_3, \rho_4)$ and the initial vector α as $\alpha = (\alpha_1, 0, \alpha_3, 0)$. The period of the solution will be $P' = 2\pi\, m_1 m_3$ while the averaging operation M has to be interpreted, when applied to the vectors involved, as $Mf = (Mf_1, 0, Mf_3, 0)$. It follows that in Eq. 3.18, function f is actually two-dimensional, i.e. $f = (f_1, 0, f_3, 0)$ and there are only two equations to be satisfied, $f_1(\alpha;\tau;\rho;\varepsilon) = 0$ and $f_2(\alpha;\tau;\rho;\varepsilon) = 0$. While in the case where all λ's are rational the solution has the form

$$y_j = \alpha_j e^{i\tau_j t} + \varepsilon\psi_j(t,\alpha,\varepsilon) \qquad j = 1, 2, 3, 4 \tag{3.23}$$

in the conditionally periodic case considered one has

$$y_j = \alpha_j e^{i\tau_j t} + \varepsilon\psi_j(t,\alpha,\varepsilon) \ \ j = 1,3 \ \ , \ \ y_k = \varepsilon\psi_k(t,\alpha,\varepsilon) \qquad k = 2,4 \tag{3.24}$$

as it is easily seen from Eq. 3.17. It is important to take into account the symmetric properties of the system. Up to now we have made no explicit use of the fact that $z_3 = \dot{z}_1$ and $z_4 = \dot{z}_2$. These relations introduce important simplifications in the actual method of computation. In fact, considering the components of Eq. 3.16, it is obvious that only two such components will need to be computed at every stage, namely, y_1 and y_2 , while y_3 and y_4 are readily computed as $y_3 = \dot{y}_1$ and $y_4 = \dot{y}_2$. For analogous reasons, the four scalar conditions corresponding to $f(\alpha;\tau;\rho;\varepsilon) = 0$ will reduce to only two since, necessarily, $\tau_3 = -\tau_1$, $\tau_4 = -\tau_2$, $\rho_3 = -\rho_1$, $\rho_4 = -\rho_2$. Also, since the solution has to be valid for any ε, $|\varepsilon| \leq \varepsilon_1$, it follows that $\alpha_3 = i\,\alpha_1\,\tau_1$ and $\alpha_4 = i\,\alpha_2\,\tau_2$.

For the actual development of the solution we consider the reduced system given in Eq. 3.6, where

$$\begin{gathered}
\varepsilon V = e^{-Bt}\varepsilon U\, e^{Bt} = e^{-Bt}(\varepsilon\,\Phi + A - B)\, e^{Bt} \\
\varepsilon V_{jk} = \sum_{l,n}(e^{-Bt})_{jl}(\varepsilon\Phi + A - B)_{jn}(e^{Bt})_{nk} = \\
= e^{(\tau_k - \tau_j)t}(\varepsilon\Phi_{jk} + A_{jk} - B_{jk}) = e^{(\tau_k - \tau_j)t}\left[\varepsilon\Phi_{jk} + (\rho_j - \tau_j)\delta_{jk}\right]
\end{gathered} \tag{3.25}$$

The successive approximations are given by

$$u_j^{(0)} = \alpha_j \ ,\ u_j^{(m)} = \alpha_j + \varepsilon\int(I - M)\sum_k V_{jk}u_k^{(m-1)}dt \ ,\ j = 1,2,3,4 \text{ and } m = 1,2,... \tag{3.26}$$

Taking into account the expression for V_{jk} given by Eq. 3.25, and considering the definition of Φ_{jk} given in section 2 of this paper, the result is

$$u_j^{(m)} = \alpha_j + \int (I-M)\left[\varepsilon(\sigma - \frac{\cos t}{1+\varepsilon\cos t})K_{jj} + \varepsilon(\rho_j - \tau_j)\right]\alpha_j\, dt + \\ -\varepsilon\sum_{k\neq j} K_{jk}\int (I-M)e^{(\tau_k-\tau_j)t}\ \frac{\cos t}{1+\varepsilon\cos t}u_k^{(m-1)}\, dt \tag{3.27}$$

For m=1, one has the equation

$$u_j^{(1)} = \alpha_j + \alpha_j\int (I-M)\left[\varepsilon(\sigma - \frac{\cos t}{1+\varepsilon\cos t})K_{jj} + \varepsilon(\rho_j - \tau_j)\right]dt + \\ -\varepsilon\sum_{k\neq j} K_{jk}\alpha_k\int (I-M)e^{(\tau_k-\tau_j)t}\ \frac{\cos t}{1+\varepsilon\cos t}dt \tag{3.28}$$

The mean value in the above expression gives

$$M\left[\varepsilon(\sigma - \frac{\cos t}{1+\varepsilon\cos t})K_{jj} + \varepsilon(\rho_j - \tau_j)\right]\alpha_j = \left[\varepsilon(\sigma-\sigma)K_{jj} + \varepsilon(\rho_j-\tau_j)\right]\alpha_j = \varepsilon(\rho_j - \tau_j)\alpha_j$$

so that $(I-M)[\cdots]\alpha_j = \varepsilon(\sigma - \frac{\cos t}{1+\varepsilon\cos t})K_{jj}\alpha_j$. The average value of $e^{(\tau_k-\tau_j)t}\cos t\,(1+\varepsilon\cos t)^{-1}$ is zero since the exponent is imaginary, i.e.

$$\tau_j - \tau_k = i(p_k/m_k - p_j/m_j) = i(p_k m_j - p_j m_k)t/m_k m_j\ .$$

The first order solution is therefore given by

$$u_j^{(1)} = \alpha_j + \varepsilon\alpha_j K_{jj}\int(\sigma - \frac{\cos t}{1+\varepsilon\cos t})dt - \varepsilon\sum_{k\neq j}\alpha_k K_{jk} I_{jk}$$

where the integral I_{jk} is given by $I_{jk} = \int e^{(\tau_k-\tau_j)t}\cos t(1+\varepsilon\cos t)^{-1}dt$

Making use of the integral (Gröbner, 1966, Integral 332.24)

$\int_0^{2\pi}\frac{\cos nx\,dx}{1+\varepsilon\cos x} = 2\pi\frac{(\sqrt{1-\varepsilon^2}-1)^n}{\varepsilon^n\sqrt{1-\varepsilon^2}},\quad n$ integer ≥ 0, one finds the Fourier Series

$$(1+\varepsilon\cos t)^{-1}\cos t = \sigma - \frac{2}{\varepsilon}\sum_{p=1}^{\infty}\sigma^p(\sqrt{1-\varepsilon^2})^{p-1}\cos pt \tag{3.29}$$

This series is convergent for any value of $|\varepsilon| < 1$. On the other end

$$\int e^{\alpha t}\cos pt\,dt = e^{\alpha t}(\alpha^2+p^2)^{-1}(\alpha\cos pt + p\sin pt) \tag{3.30}$$

and if α is imaginary the average value of $e^{\alpha t}\cos pt$ is zero. It follows for the above integral

$$I_{jk}e^{-(\tau_k-\tau_j)t} = \sigma(\tau_k-\tau_j)^{-1} + \\ -\frac{2}{\varepsilon}\sum_{p\geq 1}\sigma^p(\sqrt{1-\varepsilon^2})^{p-1}\left[(\tau_k-\tau_j)^2+p^2\right]^{-1}\left[(\tau_k-\tau_j)\cos pt + p\sin\ pt\right] \tag{3.31}$$

The final result is obtained by substituting Eq. 3.31 into the relation

$$u_j^{(1)} = \alpha_j - \frac{1}{2}\alpha_j K_{jj} \sum_{p\geq 1} p^{-1}\sigma^p(\sqrt{1-\varepsilon^2})^{p-1} sin\, pt - \sum_{k\neq j}\alpha_k K_{jk} I_{jk}$$

so that it is finally found that

$$u_j^{(1)} = \alpha_j + \varepsilon \sum_{p\geq 0} (a_{jp}^{(1)}(t)\cos pt + b_{jp}^{(1)}(t)\sin\ pt) \qquad (3.32)$$

where the time-dependent coefficients are given by Eqs. 3.33 below

$$\begin{aligned}
a_{j0}^{(1)} &= \frac{\sigma}{\varepsilon}\sum_{k\neq j}\alpha_k K_{jk}(\tau_k - \tau_j)^{-1}\exp(\tau_k - \tau_j)t \\
a_{jp}^{(1)} &= C_p^{(1)}\sum_{k\neq j}\alpha_k K_{jk}\left[(\tau_k - \tau_j)^2 + p^2\right]^{-1}(\tau_k - \tau_j)\exp(\tau_k - \tau_j)t \\
b_{jp}^{(1)} &= C_p^{(1)}\left\{\sum_{k\neq j}\alpha_k K_{jk}\left[(\tau_k - \tau_j)^2 + p^2\right]^{-1} p\exp(\tau_k - \tau_j)t + \alpha_j K_{jj}\right\} \\
C_p^{(1)} &= -\frac{2}{\varepsilon}\sigma^p(\sqrt{1-\varepsilon^2})^{p-1} \quad , \quad p = 1,2,3,...
\end{aligned} \qquad (3.33)$$

The coefficients above are actually better defined when written in the form

$$\begin{aligned}
a_{j0}^{(1)} &= \sum_{k\neq j} a_{jok}^{(1)}\exp(\tau_k - \tau_j)t\ ,\ a_{jp}^{(1)} = \sum_{k\neq j} a_{jpk}^{(1)}\exp(\tau_k - \tau_j)t \\
b_{jp}^{(1)} &= \sum_{k\neq j} b_{jpk}^{(1)}\exp(\tau_k - \tau_j)t + b_{jpj}^{(1)} \quad , \quad p = 1,2,3,...
\end{aligned} \qquad (3.34)$$

where the new coefficients are easily defined from Eq. 3.33. The infinite summation for the index p is necessary due to our inability to express the integral $I_{jk} = \int e^{(\tau_k - \tau_j)t}\cos t(1+\varepsilon\cos t)^{-1}dt$ in closed form, since the modulus of the exponent is not an integer, but the difference between two rational numbers. It is seen that the functions $u_j^{(1)}$ are infinite trigonometric series with arguments $(p_k / q_k) \pm (p_j / q_j) \pm p$ where all numbers involved are integer numbers. The infinite series show the denominators $(\tau_k - \tau_j)^2 + p^2 = -(p_k / q_k \pm p_j / q_j)^2 + p^2$ which can be zero, and therefore singular, whenever $p_k / q_k \pm p_j / q_j$ has a positive or negative integral value. The ± sign comes from the fact that $\tau_1 = -\tau_3$ and $\tau_2 = -\tau_4$. This has been discussed in section 2 of this paper.

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